## 28<sup>th</sup> International Mathematical Olympiad Havana, Cuba Day I July 10, 1987

1. Let  $p_n(k)$  be the number of permutations of the set  $\{1, \ldots, n\}$ ,  $n \ge 1$ , which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(Remark: A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if f(i) = i.)

- 2. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral AKNM and the triangle ABC have equal areas.
- 3. Let  $x_1, x_2, \ldots, x_n$  be real numbers satisfying  $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ . Prove that for every integer  $k \ge 2$  there are integers  $a_1, a_2, \ldots, a_n$ , not all 0, such that  $|a_i| \le k 1$  for all i and

$$|a_1x_1 + a_1x_2 + \dots + a_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

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- 4. Prove that there is no function f from the set of non-negative integers into itself such that f(f(n)) = n + 1987 for every n.
- 5. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.
- 6. Let n be an integer greater than or equal to 2. Prove that if  $k^2 + k + n$  is prime for all integers k such that  $0 \le k \le \sqrt{n/3}$ , then  $k^2 + k + n$  is prime for all integers k such that  $0 \le k \le n - 2$ .